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ON THE EULER'S STABILITY OF A VISCOELASTIC ROD

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The critical value of a longitudinal compressive force is calculated for a longitudinally compressed viscoelastic rod subjected to a slowly varying transverse load. A foundation of the computation method by creep according to a long-time modulus is given for the subcritical values of this force.

The method of investigation is based on some Tauberian theorems of Paley-Wiener-Gelfand type [1 – 4].

1. Let a thin viscoelastic rod of variable section of finite length l, subjected to a longitudinal compressive force P and influenced by a slowly varying external transverse load $p(x, t)$, be liable to weak bending [5]. Let us select the origin at one of the rod ends and let us assume that the rod axis is located along the x-axis at time $t = 0$. Then the deflection $y(x, t)$ of the rod axis is described by the following boundary value problem [5, 6]:

$$-\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) - P\frac{\partial^2 y}{\partial x^2} - P\int_0^t K(t,\tau)\frac{\partial^2 y}{\partial x^2}\,d\tau = -p(x,t) - \tag{1.1}$$

$$\int_0^t K(t,\tau)\,p(x,\tau)\,d\tau, \qquad 0 \leqslant x \leqslant l, \qquad 0 \leqslant t < \infty$$

$$U_i[y] = 0, \qquad i = 1, 2, 3, 4 \tag{1.2}$$

Here (1.2) are the self-adjoint boundary conditions describing the nature of the rod fastening at the ends $x = 0$ and $x = l$, $K(t, \tau)$ is the creep kernel of the rod material, $I(x)$ is the moment of inertia of a rod section with abscissa x relative to the section axis, and E is the instantaneous elastic modulus.

Let us assume that $K(t, \tau)$ is weakly singular at $0 \leqslant \tau \leqslant t < \infty$, $E = \text{const}$, $I(x)$ is finite and separated from zero in $[0, l]$ and two of the four boundary conditions (1.2) are

$$y'(0) = h_1 y(0), \qquad y'(l) = h_2 y(l), \quad 0 \leqslant |h_i| \leqslant \infty, \qquad i = 1, 2 \tag{1.3}$$

As is known [5], conditions (1.3) are satisfied for clamped, hinged, and a simply supported end.

Let us say that the vector function $f(t)$, $0 \leqslant t < \infty$ belongs to the class A, if it takes values from the Banach space $C[0, l]$ and is measurable and bounded almost everywhere in each finite interval of the half-axis $[0, \infty)$ [7].

We shall henceforth assume throughout that $p(x, t)$ belongs to A as a function of t. Then $y(x, t)$ will also belong to A.

Let $\alpha(t)$ be some continuous function positive in $[0, \infty)$.

We say that for a given P the boundary value problem (1.1), (1.2) is Euler stable with weight $\alpha(t)$ if the existence of a finite limit uniform in $x \in [0, l]$

$$L_\alpha p = \lim_{t\to\infty} p(x, t)\,\alpha(t) \tag{1.4}$$

implies the existence of a finite limit uniform in $x \in [0, l]$

$$L_\alpha y = \lim_{t\to\infty} y(x, t)\,\alpha(t) \tag{1.5}$$

We call $P_{\langle\alpha(t)\rangle}$ the critical value of the force P corresponding to the weight $\alpha(t)$ if for all $P < P_{\langle\alpha(t)\rangle}$ Euler stability with weight $\alpha(t)$ holds and for $P = P_{\langle\alpha(t)\rangle}$ it no longer takes place.

As is known [8], the investigation of the Euler stability of an elastic rod reduces to the spectral analysis of a fourth order differential operator. Investigation of the Euler stability of a viscoelastic rod would moreover require the application of some Tauberian theorems of Paley-Wiener-Gelfand type based on a study of the spectrum of Volterra operator on a half-axis [1 − 4].

This approach permitted the following results to be obtained.

Theorem 1. Let

$$1)\quad \sup_{0\le t<\infty}\int_0^t |K(t,\tau)|\,\frac{\alpha(t)}{\alpha(\tau)}\,d\tau<\infty, \qquad 2)\quad \lim_{t\to\infty}\int_\Delta K(t,\tau)\,\frac{\alpha(t)}{\alpha(\tau)}\,d\tau=0$$

for any measurable bounded set $\Delta \subset [0, \infty)$

$$3)\qquad T_k=\lim_{t\to\infty}\int_0^t K(t,\tau)\,\frac{\alpha(t)}{\alpha(\tau)}\,d\tau$$

Then the following estimate holds for the critical value of the force $P_{\langle\alpha(t)\rangle}$

$$P_{\langle\alpha(t)\rangle}\geqslant P_\varepsilon\left(1+\lim_{s\to\infty}\sup_{t\ge s}\int_s^t |K(t,\tau)|\,\frac{\alpha(t)}{\alpha(\tau)}\,d\tau\right)^{-1}$$

where P_ε is the critical Euler force for the elastic problem corresponding to the problem (1.1), (1.2).

For $P < P_{\langle\alpha(t)\rangle}$, the solution of the boundary value problem obtained from (1.1), (1.2) by replacing y by $L_\alpha y$, p by $L_\alpha p$ and the Volterra operator V with kernel $K(t, \tau)$ by the operator of multiplication by the constant T_k, is $L_\alpha y$. A more exact result (see [9]) is successfully obtained for

$$\alpha(t) = e^{-\theta t}, \qquad \operatorname{Im}\theta = 0 \tag{1.6}$$

Theorem 2. Let

$$K(t, \tau) = K_0(t-\tau) + K_1(t, \tau) \tag{1.7}$$

where

$$1)\ K_0(t) \geqslant 0, \qquad t \geqslant 0, \qquad 2)\qquad k_0(\theta)=\int_0^\infty K_0(t)\,e^{-\theta t}dt<\infty, \qquad \operatorname{Im}\theta=0$$

3) $k_0(w)$ takes on real values only on the real axis ;

4) $K_1(t, \tau)$ satisfies conditions (1), (2) of Theorem 1 for $\alpha(t) = e^{-\theta t}$;

$$5)\qquad \lim_{s\to\infty}\sup_{t\geqslant s}\int_s^t |K_1(t,\tau)|\,e^{-\theta(t-\tau)}d\tau=0$$

Then the critical force $P_{\langle e^{-\theta t}\rangle}$ is determined by the expression

$$P_{\langle e^{-\theta t}\rangle} = P_\varepsilon\,[1 + k_0(\theta)]^{-1} \tag{1.8}$$

where P_ε is the critical Euler force for the elastic problem corresponding to the problem (1.1), (1.2).

For $P < P_{\langle e^{-\theta t}\rangle}$ the limit deflection $(L_\alpha y)(x)$ is the solution of the boundary value problem obtained from (1.1), (1.2) by replacing y by $L_\alpha y$, p by $L_\alpha p$ and the operator V by the operator of multiplication by $k_0(\theta)$.

2. The set P_i, $i = 1, 2, \ldots$ of those P for which the homogeneous elastic boundary value problem corresponding to (1.1), (1.2) has a nontrivial solution will be called the Euler spectrum of the elastic problem corresponding to (1.1), (1.2). (As is known, all $P_i > 0$, and the Euler critical force P_ε coincides with P_1.)

Let $M(P)$ denote the differential operator generated by the differential expression

$$l_0[y] \equiv \left(-\frac{d^2}{dx^2}\left(EI(x)\frac{d^2}{dx^2} - PI\right)\right)y$$

and the boundary conditions (1.2), $D(M(P))$ its domain of definition in the space $C[0, l]$ and $Q_0(x, \xi, P)$ its Green's function.

Let B be the operator generated by the double differentiation operation and the boundary conditions (1.3). Its domain of definition in $C[0, l]$ will be denoted by $D(B)$. It is clear that $D(B) \supset D(M(P))$.

Let us assume $Q(P) = M^{-1}(P)B$. For fixed x and P, $Q_0(x, \xi, P) \in D(B)$, and the operator B is self-adjoint in $L_2(0, l)$, hence for any $g \in D(B)$

$$(Q(P)g)x = \int_0^l \frac{\partial^2 Q_0(x, \xi, P)}{\partial \xi^2} g(\xi)\, d\xi \tag{2.1}$$

Henceforth $Q(P)$ will everywhere be understood to be the Fredholm operator (2.1) acting in $C[0, l]$.

Let $(q_i(P))^{-1}$, $i = 1, 2, \ldots$ denote the eigenvalues of the operator $Q(P)$. We note that $q_i(0)$ agree with the Euler spectrum P_i $(i = 1, 2, \ldots)$ of the elastic problem corresponding to (1.1), (1.2) and

$$Q^{-1}(P) = B^{-1}M(P) = B^{-1}(M(0) - PB) = B^{-1}M(0) - PI = Q^{-1}(0) - PI$$

on $D(M(P))$. Hence

$$q_i(P) = P_i - P, \qquad i = 1, 2, \ldots \tag{2.2}$$

Let $\Lambda_{\langle\alpha(t)\rangle}(0, \infty)$ denote the Banach space of the functions $f(t)$ which are measurable and bounded almost everywhere in each finite interval of the half-axis $[0, \infty)$ for which the finite limit

$$L_\alpha f = \lim_{t\to\infty} f(t)\,\alpha(t) \tag{2.3}$$

with the norm

$$\|f\|_{\langle\alpha(t)\rangle} = \operatorname*{ess\,sup}_{0\leq t<\infty} |f(t)|\,\alpha(t) \tag{2.4}$$

exists, and let $Z_{\langle\alpha(t)\rangle}(0, \infty)$ be its subspace consisting of those f for which $L_\alpha f = 0$.

Analogously [3, 4], the following assertion holds.

In order that for each $f(t)$, which is measurable and bounded almost everywhere in each finite interval of the half-axis $[0, \infty)$, the existence of $L_\alpha f$ should imply the existence of $L_\alpha\varphi$, where $\varphi(t)$ is the solution of the integral equation

$$\int_0^t K(t, \tau)\,\varphi(\tau)\,d\tau - \lambda\varphi(t) = f(t) \tag{2.5}$$

it is necessary and sufficient that λ be a regular point in the space $\Lambda_{\langle\alpha(t)\rangle}(0, \infty)$ of the operator V

$$(Vf)(t) = \int_0^t K(t,\tau) f(\tau)\, d\tau \qquad (2.6)$$

We shall henceforth assume everywhere that the kernel of the Volterra operator V (2.6) satisfies the conditions (1) – (3) of Theorem 1. These conditions assure boundedness of V in both $\Lambda_{\langle\alpha(t)\rangle}(0,\infty)$ [7] and in $Z_{\langle\alpha(t)\rangle}(0,\infty)$ [3].

3. We introduce the Banach space $C\Lambda_{\langle\alpha(t)\rangle}$ of the vector functions $f(t) \in A$ for which the limit (2.3) exists, with the norm (2.4), where $|f(t_0)|$ now denotes the norm of the element $f(t_0)$ in $C[0, l]$.

Let $p(x,t)$ belong to $C\Lambda_{\langle\alpha(t)\rangle}$. The boundary value problem (1.1), (1.2) is equivalent to the following integral equation of Volterra type:

$$(PVQ(P) - I)\, y = M^{-1}(P)(I + V)\, p \qquad (3.1)$$

where V is the Volterra operator (2.6) acting in $\Lambda_{\langle\alpha(t)\rangle}(0,\infty)$.

The investigation of the Euler stability of the viscoelastic problem (1.1), (1.2) reduces to the following problem.

Find the conditions under which the solution of (3.1) also belongs to this space for $p \in C\Lambda_{\langle\alpha(t)\rangle}$ and find the expression for $L_\alpha y$ in terms of $L_\alpha p$. It is clear that if $1/P$ is a regular point of the operator $Q(P)V$ in the space $C\Lambda_{\langle\alpha(t)\rangle}$, then the boundary value problem (1.1), (1.2) is Euler stable with weight $\alpha(t)$. Hence, investigation of the Euler stability of the boundary value problem (1.1), (1.2) is closely related to the study of the spectrum of the operator $Q(P)V$ in the space $C\Lambda_{\langle\alpha(t)\rangle}$.

The theorem of multiplying spectra (see Theorem 3 in [4]) according to which (taking account of (2.2))

$$\sigma_{\langle\alpha(t)\rangle}(Q(P)V) = \bigcup_i \bigcup_{\lambda \in \sigma_{\langle\alpha(t)\rangle}(V)} \lambda \frac{1}{P_i - P} \qquad (3.2)$$

is carried over to the operator $Q(P)V$ acting in the space $C\Lambda_{\langle\alpha(t)\rangle}$. Here $\sigma_{\langle\alpha(t)\rangle}(V)$ is the spectrum of the operator V (2.6) in $\Lambda_{\langle\alpha(t)\rangle}(0,\infty)$ and $\sigma_{\langle\alpha(t)\rangle}(Q(P)V)$ is the spectrum of the operator $Q(P)V$ in $C\Lambda_{\langle\alpha(t)\rangle}$.

Lemma 1. If $1/P$ is a regular point of the operator $Q(P)V$ in $C\Lambda_{\langle\alpha(t)\rangle}$, then $L_\alpha y$ is the solution of the following boundary value problem:

$$-\frac{d^2}{dx^2}\left(EI(x)\frac{d^2 L_\alpha y}{dx^2}\right) - P(1+T_k)\frac{d^2 L_\alpha y}{dx^2} = -(1+T_k) L_\alpha p \qquad (3.3)$$

$$U_i[L_\alpha y] = 0, \qquad i = 1, 2, 3, 4$$

i.e. the problem obtained from (1.1), (1.2) by replacing y by $L_\alpha y$, p by $L_\alpha p$ and the Volterra operator V (2.6) by the operator of multiplying by the constant T_k (see condition (3) of Theorem 1).

Proof. Let us rewrite (3.1) as follows:

$$\int_0^l \frac{d^2 Q_0(x,\xi,P)}{\partial\xi^2}\left(\int_0^t K(t,\tau)\frac{\alpha(t)}{\alpha(\tau)}\left(y(\xi,\tau)\,\alpha(\tau) - (L_\alpha y)(\xi)\right) d\tau\right) d\xi +$$

$$\int_0^t K(t,\tau)\frac{\alpha(t)}{\alpha(\tau)}\, d\tau \int_0^l \frac{\partial^2 Q_0(x,\xi,P)}{\partial\xi^2}(L_\alpha y)(\xi)\, d\xi - \frac{1}{P}\, y(x,t)\,\alpha(t) =$$

$$\frac{1}{P}\left(\int_0^l Q_0(x,\xi,P)\,p(\xi,t)\,\alpha(t)\,d\xi-\int_0^l Q_0(x,\xi,P)\left(\int_0^t K(t,\tau)\frac{\alpha(t)}{\alpha(\tau)}\times\right.\right.$$

$$\left.\left.(p(\xi,\tau)\,\alpha(\tau)-(L_\alpha p)(\xi))\,d\tau\right)d\xi-\int_0^t K(t,\tau)\frac{\alpha(t)}{\alpha(\tau)}\,d\tau\int_0^l Q_0(x,\xi,P)(L_\alpha p)(\xi)\,d\xi\right.$$

and let us pass to the limit for $t\to\infty$. We obtain

$$\left(T_kQ(P)-\frac{1}{P}I\right)L_\alpha y=\frac{1}{P}(1+T_k)\,M^{-1}(P)\,L_\alpha p$$

We show that $T_k\in\sigma_{\langle\alpha(t)\rangle}(V)$. If this is not so, then (2.5) will have the solution $\varphi\in\Lambda_{\langle\alpha(t)\rangle}(0,\infty)$ for any $f\in\Lambda_{\langle\alpha(t)\rangle}(0,\infty)$. Let us choose f so that $L_\alpha f\neq 0$. But if we pass to the limit as $t\to\infty$ in (2.5), which can be rewritten as follows:

$$\int_0^t K(t,\tau)\frac{\alpha(t)}{\alpha(\tau)}(\varphi(\tau)\,\alpha(\tau)-L_\alpha\varphi)\,d\tau+\left(\int_0^t K(t,\tau)\frac{\alpha(t)}{\alpha(\tau)}\,d\tau\right)L_\alpha\varphi-T_k\varphi(t)\,\alpha(t)=f(t)\,\alpha(t)$$

then we obtain $0=L_\alpha f$.

In conformity with (3.2), $1/P\ T_k$ is a regular point of the operator $Q(P)$ in $C[0,l]$ and therefore $(T_kQ(P)-(1/P)\,I)^{-1}$ exists. Then

$$L_\alpha y=\frac{1+T_k}{P}\left(T_kQ(P)-\frac{1}{P}I\right)^{-1}M^{-1}(P)\,L_\alpha p \tag{3.4}$$

which can be rewritten as

$$L_\alpha y=-(1+T_k)\,(Q^{-1}(P)-T_kPI)^{-1}\,M^{-1}(P)\,L_\alpha p$$

It hence follows that $L_\alpha y$ is the solution of the boundary value problem (3.3).

It follows from (3.2) that the spectral radius of the operator $Q(P)\,V$ in the space $C\Lambda_{\langle\alpha(t)\rangle}$ equals the product of the spectral radius $r_{\langle\alpha(t)\rangle}(V)$ of the operator V (2.6) in $\Lambda_{\langle\alpha(t)\rangle}(0,\infty)$ by the spectral radius of the operator $Q(P)$ (2.1). The estimate found for the space $Z_{\langle\alpha(t)\rangle}(0,\infty)$ in [3] is retained for $r_{\langle\alpha(t)\rangle}(V)$ in $\Lambda_{\langle\alpha(t)\rangle}(0,\infty)$. Hence, Theorem 1 follows from Lemma 1.

4. Let us turn to the proof of Theorem 2. To do this we need the following

Lemma 2. Let $K(t,\tau)$ be representable in the form (1.7), where $K_0(t)$ satisfies condition (2) of Theorem 2, $K_1(t,\tau)$ satisfies conditions (4) and (5) of the same theorem.

Then in order that the viscoelastic problem (1.1), (1.2) be Euler stable with weight $e^{-\theta t}$, it is necessary and sufficient that

$$P_i\neq P\,(k_0(w)+1),\qquad \operatorname{Re} w\geqslant\theta \tag{4.1}$$

where P_i is the Euler spectrum of the elastic problem corresponding to (1.1), (1.2) and $k_0(w)$ is the Laplace transform of the function $K_0(t)$.

Proof. The sufficiency of condition (4.1) follows from (3.2) and from a generalized Paley-Wiener-Tauberian theorem [1] (see [3]) proved analogously [4] (see Theorem 2).

We prove its necessity. For some w with $\operatorname{Re} w\geqslant\theta$ and some P_i let the equality

$$(P_i - P) / P = k_0(w) \tag{4.2}$$

hold. This equality means that $(P_i - P)/P$ belongs to the spectrum of V in $\Lambda_{\langle e^{-\theta t}\rangle}(0, \infty)$. Hence (analogously to [4]), there exists a function $h(t) \in \Lambda_{\langle e^{-\theta t}\rangle}(0, \infty)$ such that

$$\left(\left(V - \frac{P_i - P}{P} I\right)^{-1} h\right)(t) \overline{\in} \Lambda_{\langle e^{-\theta t}\rangle}(0, \infty) \tag{4.3}$$

Let us represent (3.1) as follows:

$$y(x, t) = \left(\frac{B^{-1}}{P} p\right)(x, t) + \frac{1}{P}\left(\left(Q(P) V - \frac{I}{P}\right)^{-1}\left(Q(P) + \frac{I}{P}\right) B^{-1} p\right)(x, t)$$

By $g_i(x)$ we denote the eigenfunction of the operator $Q(P)$ (2.1) corresponding to the eigenvalue $1/(P_i - P)$ and we assume

$$p(x, t) = h(t)(Bg_i)(x) \tag{4.4}$$

Then

$$y(x, t) = \frac{1}{P}(B^{-1} g_i)(x) h(t) + \frac{P_i}{P^2(P_i - P)}\left(Q(P) V - \frac{1}{P} I\right)^{-1} g_i h\Big)(x, t) \tag{4.5}$$

Analogously to [4] we have

$$\left(\left(Q(P) V - \frac{1}{P} I\right)^{-1} g_i h\right)(x, t) = \left(\left(\frac{1}{P_i - P} V - \frac{1}{P} I\right)^{-1} h\right)(t)\, g_i(x)$$

It hence follows that taking account of (4.3) and (4.5), the deflection $y(x, t)$ does not satisfy condition (1.5) for a load of the form (4.4) (which satisfies the condition (1.4) for $\alpha(t) = e^{-\theta t}$).

Proof of Theorem 2. It easily follows from Lemma 2 that Euler stability with weight $e^{-\theta t}$, holds for $1/P > k_0(\theta)/(P_\varepsilon - P)$ and for $1/P = k_0(\theta)/(P_\varepsilon - P)$ it no longer takes place. Hence, (1.8) is obtained for $P_{\langle e^{-\theta t}\rangle}$. The second assertion will follow from Lemma 1 and from (3.2) if it can be proved that $T_k = k_0(\theta)$.

In conformity with (1.7)

$$T_k = k_0(\theta) + \lim_{t\to\infty} \int_0^t K_1(t, \tau) e^{-\theta(t-\tau)} d\tau \tag{4.6}$$

and it follows from condition (5) of Theorem 2 and condition (2) of Theorem 1 that the limit on the right in (4.6) equals zero.

Note 1. The kernel of N.Kh. Arutiunian [10]

$$K(t, \tau) = -\frac{\partial}{\partial \tau}\left(\gamma_\infty + \frac{c}{\tau}\right)(1 - e^{-\delta(t-\tau)})$$

satisfies the conditions of Theorem 2 for $\theta = 0$, since

$$\lim_{s\to\infty} \sup_{t \geqslant s} \int_s^t e^{-\delta(t-\tau)} \frac{d\tau}{\tau} < \lim_{s\to\infty} \frac{1}{s} \sup_{t \geqslant s} \int_s^t e^{-\delta(t-\tau)} d\tau = 0$$

The expression (1.8) for the critical force $P_{\langle e^{-\theta t}\rangle}$ has been obtained for the Arutiunian kernel at $\theta = 0$ in [11].

Note 2. The method of calculation $L_\alpha y$ for $\alpha(t) \equiv 1$ by using the boundary value problem (3.3) is used in engineering computations. This is the so-called method of analyzing the creep by means of the long-time modulus [6, 11]. Theorems 1 and 2 afford a foundation for this method.

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